THE TRIANGULAR FORM OF LEONTIEF'S MATRIX[1]

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Abstract

This paper presents an alternative proof of the equivalence between the existence of viable solutions to Leontief's model and the Hawkins and Simon's condition (H-S). In addition, it exposes the peculiar economic significance of the coefficients in the principal diagonal of the triangular form of Leontief's matrix, which permits us – among other things – to appreciate the tight relation between the economic and the mathematical conditions in the solution to the model. Moreover, the paper identifies some other mathematical propositions equivalent to (H-S) possessing economic interest.

Key words: Leontief, Hawkins and Simon, input–output, non-negative matrices.

Introduction

The linear systems of production equations became a relevant subject in economics after the publication of Leontief (1941), a field of research known as input–output analysis was developed upon the base of the original problems proposed in this book.[3] It is the intention of this article to present some comments related to the economic interpretation of the mathematical conditions required for a viable solution to exist in its basic model. I expose the model briefly in the first section and, after having arranged the corresponding equations system by columns, I represent it in matrix notation. This permits me to formulate the condition

[1] Translation of Benítez (2009).

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[3] Chenery and Clark (1959) describe the first 20 years of this development; ten Raa (2005) offers an actualized exposition of the uses of Leontief's model.

established by Hawkins and Simons (1949) – I will refer to it with the notation (H-S) – according to which each one of the principal minors of Leontief's matrix is greater than zero.

In the study of Leontief's model, Gauss method has been used mainly to prove that (H-S) is a necessary and sufficient condition for the existence of a unique nonnegative solution. In order to establish this result, it is enough to perform a single gaussian elimination and then proceed by mathematical induction, as in Nikaido (1970), or else to consider some properties of the whole process of triangulation, as in the article by Hawkins and Simon already cited. In the second section, I realize the complete triangulation of the equations system in order to indicate some economic aspects of the resulting system in the following one.

The proofs that I know of the equivalence between (H-S) and the existence of a nonnegative solution to the model omit to indicate how the non-zero and the zero elements of the solution are distributed among the different goods. It is natural to expect that the first values correspond to the goods produced as surplus together with those required in their production while the zeros correspond to the other goods. For this reason, in the third section, I study this distribution and I show that a Leontief's model possesses a solution with the property just mentioned if and only if it satisfies (H-S). This permits me to formulate the relation between the existence of a viable solution for Leontief's model and (H-S) in a detailed manner and also to develop an alternative proof to the fundamental result mentioned at the start of the previous paragraph.

In the fourth section, I establish the economic meaning of the coefficients in the principal diagonal of the resulting triangular matrix. The base of the interpretation is the equality between the bottom right coefficient and the quantity of the corresponding good produced as surplus after replacing the total quantity of the same good consumed in the production of one of its units. Among other things, this result permits me to express directly in mathematical terms the economic meaning of (H-S).

In the fifth section, I use the same property to formulate (H-S) more precisely than is usually done in the case of Leontief's matrices, indicating an upper bound for the values of the principal minors as well as a hierarchy in their magnitudes. Finally, in Theorems 2, 3 and 6, I establish mathematical propositions that are equivalent to (H-S) while in Lemma 1, I present a proposition that is equivalent to (H-S) in the particular case of Leontief's matrices; the economic interest is pointed out in each case. Regarding these results, it is convenient to remember that a considerable number of mathematical conditions already known are equivalent to (H-S). The most complete expositions of them that I consulted are those presented in Takayama (1985) and in Berman and Plemmons (1994).

1. Leontief's basic model

Throughout the article, I will talk about the production of a certain quantity of a good in reference to a gross production and I will specify the case of a net quantity. The economy considered integrates n industrial branches realizing simultaneously production processes of equal duration; each branch produces a particular good to which corresponds an index denoted i or j so that $i,j = 1,2,\ldots,n$. I will refer to a set $\{j1,j2,\ldots,jd,\ldots,jD\}$ as a D set if it contains D different goods.[4] For each i, the notation x_i represents the quantity of i produced in the corresponding industry and c_i the difference between this quantity and the amount of the same good that is consumed in the industrial system during the period. Also, for each pair (i,j), the technical coefficient a_{ij} represents the quantity of i that is consumed directly (in the branch producing j during the period considered) in the production of one unit of j. It is assumed that $a_{ij} \geq 0 \ \forall \ (i,j)$. There are constant returns to scale so that the coefficients are independent of the quantities produced. A good i produces a good j (not necessarily different) directly if $a_{ij} > 0$ and indirectly if there is a D set containing neither i nor j and verifying $a_{i,j1}a_{j1,j2}a_{j2,j3}\ldots a_{jD,j} > 0$.

According to the preceding definitions, the relations between the quantities consumed and produced of each good define the following equations system:

$$x_i = \Sigma_j a_{ij}x_j + c_i \qquad\qquad i = 1,2,\ldots,n \tag{1}$$

After ordering it by columns, it is possible to write this system as follows:

[4] I will refer to indexes also as goods.

$$-a_{i1}x_1 - a_{i2}x_2 \ldots - a_{i,(i-1)}x_{(i-1)} + (1-a_{ii})x_i - a_{i,(i+1)}x_{(i+1)} - \ldots$$
$$-a_{in}x_n = c_i \qquad i = 1,2,\ldots,n \tag{2}$$

To simplify, I will change the notation of the coefficients that are on the left side of the system. For each pair (i,j), let b_{ij} be equal to $-a_{ij}$ if $i = j$ and to $1 - a_{ij}$ if $i = j$; this allows me to write the preceding system as:

$$b_{i1}x_1 + b_{i2}x_2 + b_{i3}x_3 + \ldots + b_{in}x_n = c_i \qquad i = 1,2,\ldots,n \tag{3}$$

I assume that this system possesses the following properties:

$$\text{a) } 0 \le b_{ii} \ \forall\, i \qquad \text{b) } b_{ij} \le 0 \text{ if } i \neq j \qquad \text{c) } 0 \le c_i \ \forall\, i \tag{4}$$

The second inequality is due to the non-negative condition of the technical coefficients while the first and the third ones are necessary to produce at least the quantities of each good consumed, respectively, in each industry and in the set of all industries.

Introducing the $n \times n$ matrix $B = [b_{ij}]$ and the $n \times 1$ matrices $x = [x_i]$ and $c = [c_i]$, system (3) may be represented in matrix notation by means of the equation:

$$Bx = c \tag{5}$$

This formula resumes the conditions that shall be satisfied by a production program defined by x in order to produce as surplus a set of goods represented by c using the technology defined by B.[5] It is important to identify the mathematical conditions assuring the validity of the following proposition:

For any $c \geq 0$ system (5) possesses a unique solution $x \geq 0$.[6]

(6)

[5] Hawkins (1948) and Leontief (1966) arrive at an equation system similar to (5), studying the cost of production. McKenzie (1960) and Gale (1960) present several applications for this type of system.

[6] To simplify, if y is a vector, I write $y \geq 0$ to indicate that $y_i \geq 0 \ \forall\, i$ and I say that it is a non-negative vector.

With this purpose, it is useful to represent, for each j, the j-th principal minor of B with the notation D_j. It is equal to the determinant of the matrix formed by the intersection of the first j rows and the first j columns of B. The relation between these minors and the preceding proposition is as follows:

Theorem 1. *Proposition* (6) *and the following condition are equivalent:*

$$D_j > 0 \quad \forall\ j \tag{H-S}$$

The equivalence means that each proposition implies the other one.[7] The fact that a Leontief's matrix satisfies (H-S) – or any other equivalent mathematical condition – has been interpreted economically, indicating that in this case the corresponding technology is "self-sustaining", a concept defined as follows.

Definition 1. *A technology is self-sustaining if any set of industries producing a unit of a good consumes in this process, directly and indirectly (through the goods produced by the set), a total amount of less than one unit of the same good.*

This definition is based on the economic interpretations of (H-S) presented in Hawkins and Simons (1948: 248), in Takayama (1985: 361) and in Dorfman, Samuelson and Solow (1987: 215). In the fifth section, I present some results that permit to appreciate directly this interpretation.

Besides (4), system (5) also satisfies the following condition:

$$b_{ii} \leq 1 \ \forall\ i \tag{7}$$

due to the same reason justifying (4.b). Nevertheless, Theorem 1 is valid for all the systems of type (5) verifying (4) even if they do not satisfy (7). In order to identify each case, I will talk of system (5) and of matrix B to

[7] Hawkins and Simon published the first proof of this theorem in the text already cited. Almost at the same time, Georgescu-Roegen (1950) published an equivalent result that he reached independently and, in another publication, Georgescu-Roegen (1966) indicates some defects in the proof proposed by the two authors although he also defends the originality of their contribution regarding certain works published earlier.

refer to this last case and I will reserve the expressions Leontief's system and Leontief's matrix for the systems of type (5) verifying (4) and (7). For instance, system $[6]x = 1$ satisfies (H-S) but $[6]$ is not a Leontief's matrix.

2. Triangulation of system (3)

As is well known, Gauss method to solve (3) consists of eliminating the terms below the principal diagonal proceeding successively by columns starting with the first one. In order to do this, for each column *j*, the corresponding equation is divided by its coefficient in the principal diagonal (if it is not zero), then the equation is multiplied by the coefficient of coordinates *(i,j)* that is to be eliminated and the resulting equation is subtracted from equation *i*. After proceeding in the first column of the system, (3) results in:

$$\begin{array}{lllll}
b_{11}x_1 + & b_{12}x_2 & + \dots + & b_{1n}x_n & = c_1 \\
0 & (b_{22} - b_{21}b_{12}/b_{11})x_2 + & \dots + & (b_{2n} - b_{21}b_{1n}/b_{11})x_n & = c_2 - b_{21}c_1/b_{11} \\
\vdots & \vdots & & \vdots & \vdots \\
0 & (b_{n2} - b_{n1}b_{12}/b_{11})x_2 + & \dots + & (b_{nn} - b_{n1}b_{1n}/b_{11})x_n & = c_n - b_{n1}c_1/b_{11}
\end{array} \tag{8}$$

To simplify, I will introduce the super index t ($t = 1,2,\dots,n$), which indicates that, if $t = 1$, the original coefficient (identified in each case by the sub-indexes) has not been modified. If $t > 1$, the original coefficient has been modified by the linear operations required for the elimination of the terms below the principal diagonal in the first $t - 1$ columns. It is convenient to observe that, for each $t > 1$, the eliminations in column $t - 1$ affect (apart from the terms eliminated) only those coefficients whose indexes are both greater than $t - 1$. Accordingly, for each t, the coefficients b_{ij}^t and c_i^t are defined only if $t \leq (i,j)$ and, besides, if $t > 1$ it must be verified that $b_{t-1,t-1}^{t-1} \neq 0$. In this case:

$$\text{a) } b_{ij}^t = b_{ij}^{t-1} - b_{i,t-1}^{t-1}b_{t-1,j}^{t-1}/b_{t-1,t-1}^{t-1} \quad \text{b) } c_i^t = c_i^{t-1} - b_{i,t-1}^{t-1}c_{t-1}^{t-1}/b_{t-1,t-1}^{t-1} \tag{9}$$

Then, if $b_{tt}^t \neq 0$ for each $t < n$, the eliminations in the first $n - 1$ columns result in:

$$
\begin{array}{llllllll}
b_{11}^{1}x_1 + & b_{12}^{1}x_2 & + \dots + & b_{1,n-1}^{1}x_{n-1} & + & b_{1n}^{1}x_n & = c_1^{1} & \\
0 & b_{22}^{2}x_2 & + \dots + & b_{2,n-1}^{2}x_{n-1} & + & b_{2n}^{2}x_n & = c_2^{2} & \\
0 & 0 & . & . & & . & . & \\
. & . & . & . & & . & . & \\
0 & 0 & \dots & b_{n-1,n-1}^{n-1}x_{n-1} & + & b_{n-1,n}^{n-1}x_n & = c_{n-1}^{n-1} & \\
0 & 0 & \dots & 0 & + & b_{n,n}^{n}x_n & = c_n^{n} &
\end{array}
\tag{10}
$$

The coefficients of this matrix present the following relation.

Theorem 2. *The next two propositions are equivalent:*

For any t such that $t \le (i,j)$: a) $b_{ij}^{1} \ge b_{ij}^{t} > 0$ *if* $i = j$, b) $b_{ij}^{t} \le b_{ij}^{1} \le 0$ *if* $i \ne j$ *and* c) $c_i^{t} \ge c_i^{1} \ge 0 \ \forall\ i$. (11)

$$0 < b_{tt}^{t} \quad \forall\ t \tag{12}$$

Proof. Obviously, (11.a) implies (12). On the other hand, (12) together with (4.b) and (4.c) imply that (11) is also satisfied by the first equation of (10). The assumption that this is not the case for at least one of the other equations leads to a contradiction. Indeed, let t be the first equation of (10), in the natural order, for which (11) is not truth: then (11) is valid for $t-1$. As $t \le (i,j)$ we have $t-1 \ne (i,j)$ so that $b_{i,t-1}^{t-1} \le 0$ and $b_{t-1,j}^{t-1} \le 0$ according to (11.b); therefore, it follows from (9.a) that $b_{ij}^{t} \le b_{ij}^{t-1}$. This result and the fact that (11) is valid for $t-1$ imply (11.b) and the first inequality in (11.a), the second inequality of (11.a) follows from (12), proving that (11) is verified in the left side of the t-th equation. Similarly, the fact that (11) is verified for $t-1$ implies that $-b_{i,t-1}^{t-1}c_{t-1}^{t-1}/b_{t-1,t-1}^{t-1} \ge 0$ and then $c_i^{t} \ge c_i^{t-1}$ according to (9.b). This result and the fact that (11.c) is valid for $t-1$ imply that (11.c) is verified for t, contradicting the assumption that (11) was not truth and finishing the proof.

3. Non-zero and zero coordinates in the solution to (3)

Given a particular c, only those goods shall be produced that are needed to obtain as surplus exactly c. For this reason, as I already indicated, it is natural to expect that, in the solution to system (5), the quantities greater than zero correspond to the goods produced as surplus and to the goods required for their production while the zeros correspond to the rest. Nevertheless, Theorem 1 says only that there is a non-negative solution for (5) without specifying the assignment of the two types of quantities in the solution to the different goods.

In this section, I will verify that the mathematical solution corresponds to the normal economic intuition, enounced in the following proproposition.

For each $c \geq 0$ *there is one particular solution* x *to* (3) *so that* $x_i > 0$ *either if:* a) $c_i > 0$*, or* b) *i produces a good j such that* $c_j > 0$, *and* $x_i = 0$ *if:* c) *i verifies neither* a) *nor* b). (13)

In order to do this, I will now prove a result equivalent to Theorem 1, following an original procedure.

Theorem 3. *Propositions* (11) *and* (13) *are equivalent.*

Proof. I) I will prove that (13) $\Rightarrow$ (11) by induction over the index t. To this end, I will consider a particular $c > 0$ for which, according with (13.a) system (3) possesses a unique solution $x > 0$. I.a) (11) is valid for $t = 1$: (11.b) and (11.c) are verified respectively by (4.b) and (4.c). These facts and the assumptions that $c_1 > 0$ and $x > 0$ imply that $b_{11} > 0$ according to the first equation of (10), verifying (11.a). I.b) If (11) is valid for a $t - 1$ such that $1 \leq t - 1 < n$ then it is also valid for t: as $t \leq (i,j)$ we have $t - 1 \neq (i,j)$ so that $b_{i,t-1}^{t-1} \leq 0$ and $b_{t-1,j}^{t-1} \leq 0$ according to (11.b), also from (11.a) we have $b_{t-1,t-1}^{t-1} > 0$ and from (11.c) $c_{t-1}^{t-1} \geq 0$. Then, it follows from (9.a) and (9.b) respectively that $b_{ij}^{t} \leq b_{ij}^{t-1}$ and $c_i^t \geq c_i^{t-1}$. In their turn, these results and the assumption that (11) is valid for $t - 1$ imply (11.b) and (11.c) for the index t. Thus, as $x > 0$ and $c_t > 0$ we can verify by means of the t-th equation of (10) that $b_{tt}^t > 0$ validating (11.a).

II) I will prove that (11) $\Rightarrow$ (13) by induction over the index n. II.a) If $n = 1$, (11.a) $\Rightarrow b_{11} > 0$ allowing the writing of (3) in the form $x_1 = c_1/b_{11}$, which permit to verify (13) easily in this case. II.b) If (11) $\Rightarrow$ (13) in a system with $n - 1$ equations where $1 \leq n - 1$, then this occurs also in a system with n equations: if (3) verifies (11) we have $b_{11}^1 > 0$, so that the first gaussian elimination can be performed. This implies that (3) is equivalent to the system integrated by its first equation and the system formed by the remaining $n-1$ equations (S_{n-1}), that do not contain x_1, presented in (8). System (10) shows that if (3) satisfies (11) so does S_{n-1}. By hypothesis, in this case S_{n-1} verifies (13) and for this reason the proposition is satisfied by each $i > 1$. To prove that this imply that the index $i = 1$ validates (13) we can write the first equation of (3) in the form $x_1 = (c_1 - \sum_{i=2}^{n} b_{1i}x_i)/b_{11}$. Because (13) is true for S_{n-1}, we have $x_i \geq 0 \; \forall \; i > 1$. Thus, the numerator in the right side of the preceding equation is greater than zero if the first index verifies (13.a). Also if it only verifies (13.b): in this case $b_{1i} < 0$ for at least one $i > 1$ that satisfies either (13.a) or (13.b), so that $b_{1i}x_i < 0$. Consequently, in both cases $x_1 > 0$. Finally, if the first index verifies (13.c) then $b_{1i} = 0$ for each $i > 1$ validating either (13.a) or (13.b) (otherwise $i=1$ would validate 13.b). Moreover, $c_1 = 0$ and $x_i = 0$ for each $i > 1$ validating neither (13.a) nor (13.b). Consequently, the numerator just mentioned is equal to zero and so $x_1 = 0$. Therefore, (11) $\Rightarrow$ (13), ending the proof.

Propositions (6), (H-S), (11), (12) and (13) are equivalent as can be established from Theorems 1, 2, 3 and the following proposition.

Theorem 4. (H-S) *is equivalent to* (12).

Proof. The determinant of a triangular matrix is equal to the product of the coefficients in its main diagonal. Consequently, for each t, we have:

$$D_t = b_{11}{}^1 b_{22}{}^2 \ldots b_{tt}{}^t \tag{14}$$

$$\Leftrightarrow$$

$$b_{tt}{}^t = D_t / D_{t-1} \; \forall \; t > 1 \tag{15}$$

In this manner, (14) shows that (12) $\Rightarrow$ (H-S) for each t and that (H-S) $\Rightarrow$ (12) if $t = 1$, while (15) permits me to verify the last implication for each $t > 1$.[8]

[8] Gantmatcher (1960: 26) establishes that (15) is valid for every linear system of n equations with n unknowns as far as the coefficients in the principal diagonal of the triangular form of the system are all different from zero.

4. Self-sustaining technologies and condition (12)

Given a Leontief's matrix B, for each j let $c(j)$ and $x(j) = [x_{ij}]$ be two $n \times 1$ matrices. In the first one, $c_j = 1$ and $c_i = 0 \; \forall \; i \neq j$ while the second one is determined by the equation:

$$Bx(j) = c(j) \tag{16.j}$$

For each j, this matrix equation corresponds to a production program obtaining at the end exactly the same collection of goods invested at the beginning except for the quantity of j that increases in one unit. This observation allows me to formulate the following conclusions: 1) if $i \neq j$, the total quantity of i required to produce x_{jj} units of j is x_{ij} and 2) the total quantity of j required to produce x_{jj} units of j *is* $x_{jj} - 1$.

According to section 3, self-sustaining technologies verify (13). This proposition and the j-th equation of system (16. j) imply that $x_{jj} \geq 1$ and also that the equality is verified only if j does not produce itself. Then, from 2) and the assumption of constant returns to scale, it follows that, for every j, the total quantity of j required to produce one unit of j is $(x_{jj} - 1)/x_{jj} = 1 - 1/x_{jj}$. Consequently, the quantity of j produced as surplus (after discounting the total quantity of j consumed) in the production of one unit of j is determined by $1 - (1 - 1/x_{jj}) = 1/x_{jj}$. I conclude that in these technologies:

$$0 < 1/x_{jj} \leq 1 \qquad \forall \; j \tag{17}$$

Which is equivalent to:

$$0 \leq 1 - 1/x_{jj} < 1 \quad \forall \; j \tag{18}$$

The tight relation between conditions (12) and (18) in a Leontief's matrix can be appreciated in the following theorem and its corollary. It is worth mentioning that the theorem is valid for every good because the last index can be assigned to any good.[9]

[9] To reassign the indexes of the goods does not affect the solution to the system, as shown in section 1.2 of Seneta (1973).

Theorem 5. *In the triangular form of a Leontief's matrix, the bottom right coefficient is equal to the quantity of the corresponding good obtained as surplus after replacing the total quantity of this good consumed in the production of one of its units.*

Proof. I will consider the form (10) corresponding to system (16.n) in order to show that:

$$b_{nn}{}^{n} = 1/x_{nn} \tag{19}$$

With this aim, it is convenient to verify that $c_n{}^n = 1$*: if* $n = 1$, then $c_n{}^n = c(1) = 1$ and if $n > 1$, the assumption that $c_n{}^n \neq 1$ implies a contradiction. Indeed, let i be the first index in the natural order satisfying $c_i{}^t \neq c_i{}^{t-1}$ for a certain $t > 1$; according to (9.b), this is possible only if $c_{t-1}{}^{t-1} \neq 0$. However, $c_i{}^t$ is defined only if $t \leq i$; consequently $t - 1 < i$. As $t - 1 < n$, $c_{t-1}{}^{1} = 0$ so that $c_{t-1}{}^{t-1} \neq c_{t-1}{}^{1}$ in contradiction to the assumption made about i. Therefore, substituting $c_n{}^n$ for 1 in the bottom equation of the corresponding system (10) and solving for $b_{nn}{}^n$ yields (19), finishing the proof.

According to (19) and (17), if the technology is self-sustaining, then:

$$0 < b_{nn}{}^{n} \leq 1 \tag{20}$$

The equality and the strict inequality on the right side of (20) are validated respectively if n does not produce and if it produces itself. Besides, if $b_{nn}{}^n \leq 0$ in order to obtain one unit of the corresponding good, the consumption of the same good would be at least equal to *1*.

Theorem 5 permits me to interpret economically each one of the coefficients in the principal diagonal of (10). With this purpose, it is useful to consider, for each t, the equation system (21.*t*) resulting after erasing in (16.*t*) all the terms where at least one of the last $n - t$ indexes appears. This system represents an economy producing as surplus one unit of t and zero units of the first $t - 1$ goods (if $t > 1$). It is convenient to observe that the triangular form of (21.*t*) is the system remaining after erasing the last $n - t$ rows and the last $n - t$ columns in the triangular form corresponding to system (16.*t*). This means that the resulting system is similar to system (10) corresponding to (16.*n*), except for the fact that the last one has n equations instead of t. Consequently, Theorem 5 is also valid for (21.t), so that if (H-S) is satisfied, then:

$$0 < b_{tt}^{t} \leq 1 \quad \forall \quad t \tag{22}$$

The inequality on the left is due to Theorem 4. Under these conditions, according to Theorem 5, b_{tt}^{t} is equal to the quantity of t obtained as surplus after the replacement of the total quantity of this good consumed in the production of one of its units in (21.t). The difference between this quantity and the total amount of t consumed in the first t industries in (16.t) consist of the quantity of t consumed by this set of industries through the last $n – t$ goods, because they do not exist in (21.t). We arrive at the following conclusion.

Corollary to Theorem 5. *For each t, b_{tt}^{t} is equal to the quantity of t obtained as surplus in the production of one unit of t after replacing the amount of the same good consumed in the first t industries, when the quantity consumed through the last n – t industries is not included.*

This result permits me to calculate, given a set of type D, the amount of each j whose index belongs to D that is consumed to produce one unit of j in the D industries producing these goods, independently of the quantity of j consumed in the other $n – D$ industries. It is enough to assign the first D indexes to the goods in D and, successively, the D index to each one of these goods, realizing each time the corresponding triangulation. The quantity consumed in each case is equal to $1 – b_{DD}^{D}$.

Therefore, as (22) can be written in the form $1 > 1 – b_{tt}^{t} \geq 0 \ \forall \ t$, the corollary permits me to express Definition 1 in mathematical terms more directly than (H-S).

5. Another formulation of (H-S)

The following proposition presents a particular property of Leontief’s matrices.

Lemma 1. *A Leontief’s matrix verifies* (H-S) *if and only if*:

$$1 \geq D_1 \geq D_2 \ldots \geq D_n > 0 \tag{23}$$

Proof. If a Leontief's matrix validates (23), it verifies (H-S). In this case, the extreme right inequality is valid and, from (14) and (22), we may infer the first inequality (beginning from the left) in (23) while the others are established successively starting from $t = 2$ upon the base of (15) and (22), finishing the proof.

As $D_1 = 1 - a_{11}$ while (15) is verified for each $t > 1$, it follows that (23) is necessary for (22), a condition whose economic interpretation is presented in the Corollary to Theorem 5. For this reason, it may also be said that (23) is necessary in order that in the production of one unit of each good t the total quantity of the good consumed in the first t industries (without considering the consumption made in the other $n - t$ industries) is less than one unit.

The following proposition is similar to (23) but it is valid for all matrices of type B and not just for Leontief's matrices.

Theorem 6. *Let* $k = max \{1, b_{11}, b_{22},\ldots,b_{nn}\}$. (H-S) *is equivalent to*:

$$1 \geq (1/k)D_1 \geq (1/k)^2 D_2 \ldots \geq (1/k)^n D_n > 0 \tag{24}$$

Proof. As $k > 0$, (24) implies (H-S). Multiplying B by $1/k$ results in matrix $B^* = [b_{ij}^*]$, where $b_{ij}^* = (1/k)b_{ij}\ \forall\ (i,j)$. It is convenient to observe that B^* satisfies (4.a), (4.b) and (7). For each j, let D_j^* be the j-th principal minor of B^*. Because the determinant of a $j \times j$ matrix multiplied by a number $(1/k)$ is equal to the determinant of the original matrix multiplied by $(1/k)^j$, we have:[10]

$$D_j^* = (1/k)^j D_j \qquad \forall\ j \tag{25}$$

Then, if B satisfies (H-S), so does B^*. For this reason, given the other properties already mentioned, B^* satisfies the conditions of Lemma 1. Consequently, $1 \geq D_1^* \geq D_2^* \ldots \geq D_n^*$. Once the corresponding substitutions according to (25) are realized, this expression results in (24), finishing the proof.

[10] See example 1.6.4 in Goldberg (1991).

The reader may remark that (24) implies that k^j is an upper bound for the magnitude of the first j minors. Nevertheless, it follows from (17) and from the definition of k that $b_{11}b_{22}...b_{jj}$ is another upper bound not greater than k^j.

6. Conclusions

The article does not take into account certain aspects of the economic activities – like their ecological dimension – whose inclusion in the analysis may eventually lead to the imposition of some restrictions on the results. Nevertheless, the exclusive focus on Leontief's basic model is justified, from my point of view, inasmuch as the results obtained have economic interest. In this section, I am going to add to the arguments already exposed a few comments that may be useful in order to clarify this point.

Given the fact that every production process employs labor, together with the goods consumed as inputs, condition (6) characterizes a set of technologies that I find it appropriate to call viable. It contains all those technologies that permit any given surplus to be obtained after the replacement of the goods consumed as inputs. Definition 1 indicates a necessary and sufficient condition for a technology to belong to this set, which also permits us to designate the technology as "self-sustainable". When we refer to the industrial system as a whole, the meaning of this definition is stated in (18): the total quantity of each good consumed in the production of 1 unit of the same good is less than 1 unit.

To recognize the necessity of the condition established in Definition 1 for a technology to be viable, it is enough to consider the case just mentioned. If a certain good does not fulfill the condition, then it is not possible to produce it as surplus, replacing at the same time the total quantity of the same good consumed in the process. The sufficiency of the condition, proved in Theorem 1, may be easier to grasp with the aid of the Corollary to Theorem 5 because it permits us to appreciate the economic meaning of a mathematical condition equivalent to (H-S).

On the other hand, the article develops an original proof to the equivalence between condition (H-S) and the "self-sustainability" of a Leontief's technology which provides an alternative way to study these technologies. Compared to the procedures already published (among which the most widely cited version appears in the book previously mentioned by Nikaido)[11] it presents the following advantages: a) the mathematical tools employed are simpler because the proof is based on Gauss method, b) the result contains more information because it permits to identify the distribution of the zero and the greater than zero coordinates in the solution and c) the triangular form of Leontief's model permits to present the economic meaning of the mathematical conditions involved more directly.

As already pointed out in the introduction, I have also added to those already known two mathematical conditions equivalent to (H-S) that – to the best of my knowledge – have not been published before this work. Moreover, I identify two other conditions equivalent to (H-S) in the particular case of Leontief's matrices. In my opinion, it is convenient to publish them because they permit us to appreciate some mathematical aspects of Leontief's model and due to the economic implications indicated in each case.

Finally, the definition of the economic meaning of the coefficients in the principal diagonal of the triangular form of a Leontief's matrix may facilitate some comparative studies with other disciplines. To illustrate this point, I refer the reader to the interpretation of the same coefficients made by Gantmatcher (1960: 28–31).

[11] For instance, in Takayama (1985: 359), also in Uribe (1997: 104).

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